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Note

Directed triangles in directed graphs

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Abstract

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We show that each directed graph on n vertices, each with indegree and outdegree at least n/t, where $t = 5 - \sqrt{5} + \frac{1}{2}\sqrt{47 - 21\sqrt{5}} = 2.8670975 \cdots$, contains a directed circuit of length at most 3.

It is an intriguing conjecture of Caccetta and Haggkvist [1] that any directed graph on n vertices, each with outdegree at least k, contains a directed circuit of length at most $\lceil n/k \rceil$. (In this paper, directed graphs have no loops and no parallel arcs (in the same or the opposite direction).)

A particularly interesting special case that is still open is: any directed graph on n vertices with minimum outdegree at least $n/3$ has a directed triangle. The best result along these lines is proved in [1]: any directed graph on n vertices with

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minimum outdegree at least s, where

$$s := \tfrac{3}{2} + \tfrac{1}{2}\sqrt{5} = 2.618034\cdots, \tag{1}$$

contains a directed triangle.

It is not even known whether any directed graph on n vertices, each with both indegree and outdegree equal to $n/3$, contains a directed triangle.

In this note we use the result of [1] to show the following.

Theorem. *Any directed graph on n vertices, each with both indegree and outdegree at least n/t, where*

$$t := 5 - \sqrt{5} + \tfrac{1}{2}\sqrt{47 - 21\sqrt{5}} = 2.8670975\cdots, \tag{2}$$

contains a directed triangle.

Proof. Suppose $D = (V, A)$ is a directed graph with $|V| = n$, with each indegree and each outdegree at least n/t, and without any directed triangle. Let $k := \lceil n/t \rceil$. We may assume

$$5 - \sqrt{5} - \frac{1}{2}\sqrt{47 - 21\sqrt{5}} \leqslant \frac{n}{k} \leqslant 5 - \sqrt{5} + \frac{1}{2}\sqrt{47 - 21\sqrt{5}}. \tag{3}$$

(We can replace any vertex v of D by l pairwise non-adjacent vertices, and any arc (u, v) by l^2 arcs, from each of the l copies of u to each of the l copies of v. We obtain a directed graph D' with $n' := nl$ vertices, such that each vertex has indegree and outdegree at least n'/t, and such that D' has no directed triangle. By choosing l large enough, $n'/k = n'/\lceil n'/t \rceil$ will satisfy (3).)

Assume that deleting any arc would give a vertex of indegree or outdegree less than k. We show:

$$\text{there exists a vertex } v' \text{ with both indegree and outdegree equal to } k. \tag{4}$$

Suppose such a vertex does not exist. Let W be the set of vertices of indegree equal to k. Then there are no arcs leaving W (since any such arc could be deleted without violating the condition that each indegree and each outdegree is at least k). Since W contains at most $k\,|W|$ arcs, it follows that if $W \neq \emptyset$, W contains a vertex of outdegree at most k. If $W = \emptyset$, we apply this argument to the set of vertices of outdegree equal to k (which set should be nonempty if $W = \emptyset$).

For each $v \in V$ let E_v^+ and E_v^- denote the sets of outneighbours and inneighbours of v, respectively. For $u, v, w \in V$ let

$$E_{uv}^+ := E_u^+ \cap E_v^+, \qquad E_{uv}^- := E_u^- \cap E_v^-,$$
$$E_{uvw}^+ := E_u^+ \cap E_v^+ \cap E_w^+, \quad \text{and} \quad E_{uvw}^- := E_u^- \cap E_v^- \cap E_w^-.$$

Moreover let

$$\varepsilon_v^+ := |E_v^+|, \qquad \varepsilon_v^- := |E_v^-|, \qquad \varepsilon_{uv}^+ := |E_{uv}^+|,$$
$$\varepsilon_{uv}^- := |E_{uv}^-|, \qquad \varepsilon_{uvw}^+ := |E_{uvw}^+| \quad \text{and} \quad \varepsilon_{uvw}^- := |E_{uvw}^-|.$$

We observe that for all $u, v, w \in V$:

$$\text{if } (u, v), (v, w), (u, w) \in A$$
$$\text{then } \varepsilon_{uv}^{-} + \varepsilon_{vw}^{+} \geqslant \varepsilon_u^{-} + \varepsilon_v^{-} + \varepsilon_v^{+} + \varepsilon_w^{+} - n \geqslant 4k - n. \tag{5}$$

Indeed, as D has no directed triangles, $(E_u^- \cup E_v^-) \cap (E_v^+ \cup E_w^+) = \emptyset$. So $|E_u^- \cup E_v^-| + |E_v^+ \cup E_w^+| \leqslant n$. Now

$$\varepsilon_{uv}^{-} = |E_{uv}^{-}| = |E_u^- \cap E_v^-| = |E_u^-| + |E_v^-| - |E_u^- \cup E_v^-| = \varepsilon_u^- + \varepsilon_v^- - |E_u^- \cup E_v^-|\,.$$

Similarly, $\varepsilon_{vw}^{+} = \varepsilon_v^+ + \varepsilon_w^+ - |E_v^+ \cup E_w^+|$. This gives the first inequality in (5). The second inequality follows from the assumption that each indegree and each outdegree is at least k.

We next show:

$$\text{for each arc } (u, v) \text{ of } D\text{: } \varepsilon_{uv}^{-} \geqslant (3k - n)s \text{ and } \varepsilon_{uv}^{+} \geqslant (3k - n)s, \tag{6}$$

where s is as defined in (1).

To prove this, we may assume by symmetry that $\varepsilon_{uv}^{+} \geqslant \varepsilon_{uv}^{-}$. First we show $\varepsilon_{uv}^{-} > 0$, i.e., $E_{uv}^{-} \neq \emptyset$. If E_{uv}^{-} would be empty, then $E_v^- \cup E_v^+ \subseteq V \setminus E_u^-$, since there is no directed triangle. Hence $|E_v^- \cup E_v^+| \leqslant n - k$. As $|E_v^-| \geqslant k$ and $|E_v^+| \geqslant k$ and as $n/k \leqslant t < 3$, we know $E_v^- \cap E_v^+ \neq \emptyset$, implying that there is a directed digon, contradicting our assumption.

Applying Caccetta and Haggkvist's result [1] to the subgraph induced by $E_{uv}^{+} \neq \emptyset$ we obtain the existence of a $w \in E_{uv}^{+}$ so that $\varepsilon_{uvw}^{+} < \varepsilon_{uv}^{+}/s$. By (5):

$$\varepsilon_{uv}^{-} \geqslant \varepsilon_u^- + \varepsilon_v^- + \varepsilon_v^+ + \varepsilon_w^+ - n - \varepsilon_{vw}^{+} \geqslant 3k - n + \varepsilon_v^+ - \varepsilon_{vw}^{+}. \tag{7}$$

Since $\varepsilon_{uvw}^{+} + \varepsilon_v^+ \geqslant |E_{uv}^{+} \cap E_{vw}^{+}| + |E_{uv}^{+} \cup E_{vw}^{+}| = \varepsilon_{uv}^{+} + \varepsilon_{vw}^{+}$, (7) implies

$$\begin{aligned}\varepsilon_{uv}^{-} &\geqslant 3k - n + \varepsilon_{uv}^{+} - \varepsilon_{uvw}^{+} > 3k - n + (1 - s^{-1})\varepsilon_{uv}^{+} \\ &\geqslant 3k - n + (1 - s^{-1})\varepsilon_{uv}^{-}.\end{aligned} \tag{8}$$

This implies (6).

Now consider vertex v' described in (4). Since the subgraph induced by $E_{v'}^{-}$ contains no loops or directed digons, the number of arcs contained in $E_{v'}^{-}$ is at most $\varepsilon_{v'}^{-}(\varepsilon_{v'}^{-} - 1)/2 < \frac{1}{2}k^2$. That is,

$$\sum_{u \in E_{v'}^{-}} \varepsilon_{uv'}^{-} < \tfrac{1}{2}k^2. \tag{9}$$

Similarly,

$$\sum_{w \in E_{v'}^{+}} \varepsilon_{v'w}^{+} < \tfrac{1}{2}k^2. \tag{10}$$

Let u' be a vertex of minimum indegree in the subgraph induced by $E_{v'}^{-}$ and let w' be a vertex of minimum outdegree in the subgraph induced by $E_{v'}^{+}$. So $\varepsilon_{u'v'}^{-} \leqslant \varepsilon_{uv'}^{-}$ for all $u \in E_{v'}^{-}$ and $\varepsilon_{v'w'}^{+} \leqslant \varepsilon_{v'w}^{+}$ for all $w \in E_{v'}^{+}$.

First assume

$$\varepsilon_{u'v'}^{-} + \varepsilon_{v'w'}^{+} > 4k - n. \tag{11}$$

Then (9) and (10) imply $k^2 > (4k-n)k$, i.e., $n/k > 3$, a contradiction. So we know

$$\varepsilon^-_{u'v'} + \varepsilon^+_{v'w'} \leqslant 4k - n. \tag{12}$$

On the other hand, by (5) we know that for all $w \in E^+_{u'v'}$ one has $\varepsilon^-_{u'v'} + \varepsilon^+_{v'w} \geqslant 4k - n$. This gives:

$$\begin{aligned} \sum_{w \in E^+_{v'}} \varepsilon^+_{v'w} &= \sum_{w \in E^+_{u'v'}} \varepsilon^+_{v'w} + \sum_{w \in E^+_{v'} \setminus E^+_{u'v'}} \varepsilon^+_{v'w} \\ &\geqslant \varepsilon^+_{u'v'}(4k - n - \varepsilon^-_{u'v'}) + (\varepsilon^+_{v'} - \varepsilon^+_{u'v'})\varepsilon^+_{v'w'}. \end{aligned} \tag{13}$$

Similarly:

$$\sum_{u \in E^-_{v'}} \varepsilon^-_{uv'} \geqslant \varepsilon^-_{v'w'}(4k - n - \varepsilon^+_{v'w'}) + (\varepsilon^-_{v'} - \varepsilon^-_{v'w'})\varepsilon^-_{u'v'}. \tag{14}$$

Combining (9), (10), (13) and (14) gives:

$$\begin{aligned} k^2 &> \varepsilon^+_{u'v'}(4k - n - \varepsilon^-_{u'v'}) + (\varepsilon^+_{v'} - \varepsilon^+_{u'v'})\varepsilon^+_{v'w'} + \varepsilon^-_{v'w'}(4k - n - \varepsilon^+_{v'w'}) \\ &\quad + (\varepsilon^-_{v'} - \varepsilon^-_{v'w'})\varepsilon^-_{u'v'} \\ &= \varepsilon^-_{v'}\varepsilon^-_{u'v'} + \varepsilon^+_{v'}\varepsilon^+_{v'w'} + (\varepsilon^+_{u'v'} + \varepsilon^-_{v'w'})(4k - n - \varepsilon^-_{u'v'} - \varepsilon^+_{v'w'}) \\ &\geqslant k(\varepsilon^-_{u'v'} + \varepsilon^+_{v'w'}) + 2(3k-n)s(4k - n - \varepsilon^-_{u'v'} - \varepsilon^+_{v'w'}) \\ &= 2(3k-n)(4k-n)s + (k - 2(3k-n)s)(\varepsilon^-_{u'v'} + \varepsilon^+_{v'w'}) \\ &\geqslant 2(3k-n)(4k-n)s + (k - 2(3k-n)s) \cdot 2(3k-n)s \\ &= 2(3k-n)(5k - n - 2(3k-n)s)s. \end{aligned}$$

So

$$(4s^2 - 2s)(n/k)^2 - (24s^2 - 16s)(n/k) + (36s^2 - 20s + 1) > 0, \tag{16}$$

i.e.,

$$(11 + 5\sqrt{5})(n/k)^2 - (60 + 28\sqrt{5})(n/k) + (82 + 39\sqrt{5}) > 0. \tag{17}$$

This contradicts (3). □

Acknowledgements

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References

[1] L. Caccetta and R. Haggkvist, On minimal digraphs with given girth, in: F. Hoffman et al., eds, Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congr. Numer. 21 (Utilitas Math., Winnipeg, 1978) 181–187.